

HARISH-CHANDRA MODULES OVER THE \mathbb{Q} HEISENBERG-VIRASORO ALGEBRA

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ABSTRACT. In this paper, it is proved that all irreducible Harish-Chandra modules over the \mathbb{Q} Heisenberg-Virasoro algebra are of intermediate series (all weight spaces are 1-dimensional).

1. Introduction.

The Heisenberg-Virasoro algebra is the universal central extension of the Lie algebra of differential operators on a circle of order at most one. It contains the classical Heisenberg algebra and the Virasoro algebra as subalgebras. The structure of their irreducible highest weight modules was studied in [ACKP, B1]. Irreducible Harish-Chandra modules over the Heisenberg-Virasoro algebra were totally classified in [LZ]. They are either highest weight modules, lowest weight modules, or modules of intermediate series. The representation theory of the Heisenberg-Virasoro algebra is closely related to those of other Lie algebras, such as the Virasoro algebra and toroidal Lie algebras, see [B2, FO, JJ]. For other results on generalized Heisenberg-Virasoro algebras, please see [FO, SS, SJ] and references therein.

Recently, some authors introduced generalized Heisenberg-Virasoro algebras and started to study their representations (see [LJ, SS]). In the present paper, we give the classification of irreducible Harish-Chandra modules over the \mathbb{Q} Heisenberg-Virasoro algebra. Similar to the \mathbb{Q} Virasoro algebra case [Ma], they are only modules of intermediate series. The main idea in our proof (Lemma 3.1) is similar to those used in [Ma] and [GLZ1].

In this paper we always denote by $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$ the sets of integers, rational and complex numbers, respectively. Now we give the definitions of generalized Heisenberg-Virasoro algebras and the \mathbb{Q} Heisenberg-Virasoro algebra.

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Definition 1.1. Suppose that G is an additive subgroup of \mathbb{C} . The generalized Heisenberg-Virasoro algebra $\text{HVir}[G]$ is a Lie algebra over \mathbb{C} with a basis:

$$\{d_g, I(g), C_D, C_{DI}, C_I | g \in G\},$$

subject to the Lie brackets given by:

$$[d_g, d_h] = (h - g)d_{g+h} + \delta_{g,-h} \frac{g^3 - g}{12} C_D, \quad (1.1)$$

$$[d_g, I(h)] = hI(g + h) + \delta_{g,-h}(g^2 + g)C_{DI}, \quad (1.2)$$

$$[I(g), I(h)] = g\delta_{g,-h}C_I, \quad (1.3)$$

$$[\text{HVir}[G], C_D] = [\text{HVir}[G], C_{DI}] = [\text{HVir}[G], C_I] = 0. \quad (1.4)$$

It is easy to see that $\text{HVir}[G] \simeq \text{HVir}[G']$ iff there exists nonzero $a \in \mathbb{C}$ such that $G = aG'$. When $G = \mathbb{Z}$, $\text{HVir}[\mathbb{Z}] = \text{HVir}$ is the classical Heisenberg-Virasoro algebra. When $G = \mathbb{Q}$, we call $\text{HVir}[\mathbb{Q}]$ the \mathbb{Q} *Heisenberg-Virasoro algebra*, which is the main object of this paper. An $\text{HVir}[G]$ module V is said to be trivial if $\text{HVir}[G]V = 0$, and we denote the 1-dimensional trivial module by T .

The modules of intermediate series $V(\alpha, \beta; F)$ over $\text{HVir}[G]$ are defined as follows for every $\alpha, \beta, F \in \mathbb{C}$. As a vector space over \mathbb{C} , $V(\alpha, \beta; F)$ has a basis $\{v_g | g \in G\}$ and the actions are:

$$d_g v_h = (\alpha + h + g\beta)v_{g+h}, \quad I(g)v_h = Fv_{g+h}, \quad (1.5)$$

$$C_D v_g = 0, \quad C_I v_g = 0, \quad C_{DI} v_g = 0, \quad \forall g, h \in G. \quad (1.6)$$

It is well known that $V(\alpha, \beta; F) \cong V(\alpha + g, \beta; F)$ for any $\alpha, \beta, F \in \mathbb{C}$ and $g \in G$. So we always assume that $\alpha = 0$ when $\alpha \in G$. It is also easy to see that $V(\alpha, \beta; F)$ is reducible if and only if $F = 0, \alpha = 0$, and $\beta \in \{0, 1\}$. The module $V(0, 0; 0)$ has a 1-dimensional submodule T and $V(0, 0; 0)/T$ is irreducible; $V(0, 1; 0)$ has codimension 1 irreducible submodule. We denote the unique nontrivial irreducible sub-quotient of $V(\alpha, \beta; F)$ by $V'(\alpha, \beta; F)$. It is easy to see that $V'(0, 0; 0) \cong V'(0, 1; 0)$.

Remark. It is not hard to verify that, for $F \neq 0$, $V(\alpha, \beta; F) \cong V(\alpha', \beta'; F')$ iff $\alpha - \alpha' \in G$ and $(\beta, F) = (\beta', F')$.

Our main theorem is the following:

Theorem 1.2. *Suppose that V is an irreducible nontrivial Harish-Chandra module over $\text{HVir}[\mathbb{Q}]$. Then V is isomorphic to $V'(\alpha, \beta; F)$ for suitable $\alpha, \beta, F \in \mathbb{C}$.*

The paper is organized as follows. In Section 2, we collect some known results on the Virasoro algebra and on the Heisenberg-Virasoro algebra for later use. In section 3, we give the proof of the main theorem.

2. Preliminaries.

Since generalized Heisenberg-Virasoro algebras are closely related to generalized Virasoro algebras, we first recall some results on generalized Virasoro algebras.

Definition 2.1 Let G be a nonzero additive subgroup of \mathbb{C} . The generalized Virasoro algebra $\text{Vir}[G]$ is a Lie algebra over \mathbb{C} with a basis: $\{d_g, C_D | g \in G\}$ and subject to the Lie brackets given by:

$$[d_g, d_h] = (h - g)d_{g+h} + \delta_{g,-h} \frac{g^3 - g}{12} C_D, \quad [\text{Vir}[G], C_D] = 0. \quad (2.1)$$

Notice that $\text{Vir}[G]$ is a subalgebra of $\text{HVir}[G]$. When $G = \mathbb{Z}$, $\text{Vir}[\mathbb{Z}] = \text{Vir}$ is the classical Virasoro algebra and when $G = \mathbb{Q}$, $\text{Vir}[\mathbb{Q}]$ is called the \mathbb{Q} Virasoro algebra.

A module of intermediate series $V(\alpha, \beta)$ has a \mathbb{C} -basis $\{v_g | g \in G\}$ and the Vir-actions are:

$$d_g v_h = (\alpha + h + g\beta)v_{g+h}, \quad C_D v_g = 0, \quad \forall g, h \in G. \quad (2.2)$$

It is well known that $V(\alpha, \beta)$ is reducible if and only if $\alpha \in G$ and $\beta \in \{0, 1\}$. The unique nontrivial irreducible sub-quotient of $V(\alpha, \beta)$ is denoted by $V'(\alpha, \beta)$. It is also well known that $V(\alpha, \beta) \cong V(\alpha + g, \beta)$ for any $\alpha, \beta \in \mathbb{C}$ and any $g \in G$. So we always assume that $\alpha = 0$ if $\alpha \in G$.

Mazorchuk [Ma] classified irreducible Harish-Chandra modules over $\text{Vir}[\mathbb{Q}]$:

Theorem 2.2. *Any irreducible Harish-Chandra module over $\text{Vir}[\mathbb{Q}]$ is isomorphic to $V'(\alpha, \beta)$ for suitable $\alpha, \beta \in \mathbb{C}$.*

Now we consider the \mathbb{Q} Heisenberg-Virasoro algebra $\text{HVir}[\mathbb{Q}]$, which contains a classical Heisenberg-Virasoro algebra $\text{HVir}[\mathbb{Z}] = \text{HVir}$. The classification of irreducible Harish-Chandra modules over HVir was given in [LZ]:

Theorem 2.3. *Any irreducible Harish-Chandra module over HVir is isomorphic to either a highest weight module, a lowest weight module, or $V'(\alpha, \beta; F)$ for suitable $\alpha, \beta, F \in \mathbb{C}$.*

3. Proof of Theorem 1.2.

We decompose \mathbb{Q} as the union of a series of rings $\mathbb{Q}_k = \{\frac{n}{k!} | n \in \mathbb{Z}\}$, $k \in \mathbb{N}$. Then we can view each $\text{HVir}[\mathbb{Q}_k]$ as a subalgebra of $\text{HVir}[\mathbb{Q}]$ naturally, and thus $\text{HVir}[\mathbb{Q}] = \bigcup_{k \in \mathbb{N}} \text{HVir}[\mathbb{Q}_k]$. Clearly, $\text{HVir}[\mathbb{Q}_k] \simeq \text{HVir}$ for any $k \in \mathbb{N}$. For convenience, we write $U(\mathbb{Q}) = U(\text{HVir}[\mathbb{Q}])$ and $U(\mathbb{Q}_k) = U(\text{HVir}[\mathbb{Q}_k])$ for short. Denote $U(\mathbb{Q})_q = \{u \in U(\mathbb{Q}) | [d_0, u] = qu\}$ and $U(\mathbb{Q}_k)_q = \{u \in U(\mathbb{Q}_k) | [d_0, u] = qu\}$, for any $q \in \mathbb{Q}$.

Lemma 3.1. *Suppose that N is a finite dimensional irreducible $U(\mathbb{Q})_0$ -module, then there is some $k \in \mathbb{N}$ such that N is an irreducible $U(\mathbb{Q}_k)_0$ -module.*

Proof. There is an associative algebra homomorphism $\Phi: U(\mathbb{Q})_0 \rightarrow gl(N)$, where $gl(N)$ is the general linear associative algebra of N .

Since N and hence $gl(N)$ are both finite dimensional, then $U(\mathbb{Q})_0 / \ker(\Phi)$ is finite dimensional. Take $y_1, y_2, \dots, y_m \in U(\mathbb{Q})_0$ such that $\overline{y_1}, \overline{y_2}, \dots, \overline{y_m}$ become a basis of $U(\mathbb{Q})_0 / \ker(\Phi)$. Then there is some $k \in \mathbb{N}$ such that y_1, y_2, \dots, y_m are all in $U(\mathbb{Q}_k)_0$.

We claim that N is an irreducible $U(\mathbb{Q}_k)_0$ -module. Let M be a proper $U(\mathbb{Q}_k)_0$ -submodule of N . For any $y \in U(\mathbb{Q})_0$, there is some $y_0 \in \ker(\Phi)$ and $a_i \in \mathbb{C}$, such that $y = y_0 + \sum_{i=1}^m a_i y_i$. Thus $yM = (y_0 + \sum_{i=1}^m a_i y_i)M \subset (\sum a_i y_i)M \subset M$, that is, M is a $U(\mathbb{Q})_0$ -submodule of N , forcing $M = 0$. Thus N is irreducible over $U(\mathbb{Q}_k)_0$. \square

Now we fix a nontrivial irreducible Harish-Chandra module V over $HVir[\mathbb{Q}]$. Then, there exists some $\alpha \in \mathbb{C}$ such that $V = \bigoplus_{q \in \mathbb{Q}} V_q$, where $V_q = \{v \in V \mid d_0 v = (\alpha + q)v\}$. We define the support of V as $\text{supp } V = \{q \in \mathbb{Q} \mid V_q \neq 0\}$.

Lemma 3.2. *$\text{supp } V = \mathbb{Q}$ or $\mathbb{Q} \setminus \{-\alpha\}$ and $\dim V_q = 1, \forall q \in \text{supp } V$.*

Proof. First view V as a $Vir[\mathbb{Q}]$ module, by Theorem 2.2, then $\dim V_p = \dim V_q, \forall p, q \in \mathbb{Q} \setminus \{-\alpha\}$. Particularly, V is a uniformly bounded module, i.e., the dimensions of all weight spaces are bounded by a positive integer.

Suppose that $\dim V_q > 1$ for some $q \in \text{supp } V$. Then it is easy to see that V_q is an irreducible $U[\mathbb{Q}]_0$ -module. By Lemma 3.1, there is some $k \in \mathbb{N}$ such that V_q is an irreducible $U(\mathbb{Q}_k)_0$ -module.

We consider the $HVir[\mathbb{Q}_k]$ -module $W = U(\mathbb{Q}_k)V_q$, which is uniformly bounded. Then by Theorem 2.3, there is a composition series of $HVir[\mathbb{Q}_k]$ -modules:

$$0 = W^{(0)} \subset W^{(1)} \subset W^{(2)} \subset \dots W^{(m)} = W.$$

Each factor $V^{(i)}/V^{(i-1)}$ is either trivial or of intermediate series, and in both cases all nonzero weight spaces are 1-dimensional.

Take the first $V^{(i)}$ such that $\dim V_q^{(i)} \neq 0$. Then $\dim V_q^{(i)} = \dim(V^{(i)}/V^{(i-1)})_q = 1$. But on the other hand, $V_q^{(i)}$ is a $U(\mathbb{Q}_k)_0$ -module, i.e., a nontrivial proper $U(\mathbb{Q}_k)_0$ -submodule of V_q , contradiction. Thus $\dim V_q = 1, \forall q \in \text{supp } V$.

Since V is nontrivial, we must have some $q \in \text{supp } V \setminus \{-\alpha\}$. Our result follows since $\dim V_p = \dim V_q = 1, \forall p \in \mathbb{Q} \setminus \{-\alpha\}$. \square

Now we can give the proof of our main theorem:

Proof of Theorem 1.2. By Lemma 3.2, we know that $\text{supp } V = \mathbb{Q}$ for some $\alpha \in \mathbb{C}$ or $\text{supp } V = \mathbb{Q} \setminus \{-\alpha\}$. Denote $V^{(0)} = 0$ and $V^{(k)} = \bigoplus_{q \in \mathbb{Q}_k} V_q$ for all $k \geq 1$. Then

$\text{supp } V^{(k)} = \{q \in \mathbb{Q} \mid V_q^{(k)} \neq 0\} = \text{supp } V \cap \mathbb{Q}_k$ for all $k \geq 1$. We have a vector space filtration of V :

$$0 = V^{(0)} \subset V^{(1)} \subset V^{(2)} \subset \dots \subset V^{(k)} \subset \dots \subset V \text{ and } V = \bigcup_{k=0}^{\infty} V^{(k)}.$$

It is clear that each $V^{(k)}$ can be viewed as an $\text{HVir}[\mathbb{Q}_k]$ -module, and each $\text{HVir}[\mathbb{Q}_k]$ is isomorphic to HVir .

If $-\alpha \notin \text{supp } V$, then each $V^{(k)}$ is irreducible over $\text{HVir}[\mathbb{Q}_k]$ for any $k \in \mathbb{Z}$ by Lemma 3.2 and Theorem 2.3.

If $-\alpha \in \text{supp } V$, we have assumed that $\alpha = 0$ in this case. Then we have $u_p \in U(\text{HVir}[\mathbb{Q}])_p$ and $u_q \in U(\text{HVir}[\mathbb{Q}])_q$ for some $0 \neq p, q \in \mathbb{Q}$ such that $u_p V_0 = V_p$ and $u_q V_{-q} = V_0$, since V is an irreducible $\text{HVir}[\mathbb{Q}]$ -module. Then there is some k_0 such that $u_p, u_q \in U(\text{HVir}[\mathbb{Q}_{k_0}])$, thus $V^{(k_0)}$ is irreducible as an $\text{HVir}[\mathbb{Q}_{k_0}]$ -module.

Since $V^{(k_0)}$ is irreducible over $\text{HVir}[\mathbb{Q}_{k_0}]$, then $V^{(k)}$ is irreducible over $\text{HVir}[\mathbb{Q}_k]$ for any $k \geq k_0$. By Theorem 2.3, we see that $C_D V = C_I V = C_{DI} V = 0$, that $I(0)$ acts as a scalar $F \in \mathbb{C}$, and that there are some $\alpha_k, \beta_k \in \mathbb{C}$ such that $V^{(k)} \cong V'(\alpha_k, \beta_k; F)$ as modules over $\text{HVir}[\mathbb{Q}_k]$ with $k \geq k_0$. Write $\alpha = \alpha_{k_0}$ and $\beta = \beta_{k_0}$ for short.

If $F = 0$, V is an irreducible $\text{Vir}(\mathbb{Q})$ -module and $I(p)V = 0$ for any $p \in \mathbb{Q}$. Theorem 2.1 follows from Theorem 2.2. Next we assume that $F \neq 0$.

Now we need to prove that $V \cong V'(\alpha, \beta; F)$ as modules over $\text{HVir}[\mathbb{Q}]$, i.e., we can choose a basis of V such that (1.5) and (1.6) hold. We proceed by choosing a basis of each $V^{(k)}$ inductively, such that (1.5) and (1.6) hold when one only consider the actions of $\text{HVir}[\mathbb{Q}_k]$, and that the basis of each $V^{(k+1)}$ is the extension of the basis of $V^{(k)}$ for any $k \geq k_0$.

Naturally, we have such a basis for $V^{(k_0)}$. Now suppose that we have such bases for $V^{(k)}, \forall k < m$ for some $m > k_0$. That is, we have a basis $\{v_q \in V_q \mid q \in \text{supp } V^{(m-1)}\}$ such that $d_p v_q = (\alpha + q + p\beta)v_{p+q}$ and $I(p)v_q = Fv_{p+q}, p, q \in \text{supp } V^{(m-1)}$. Now we consider $V^{(m)}$.

Let Φ be the canonical isomorphism $\text{HVir} \rightarrow \text{HVir}[\mathbb{Q}_m]$, defined by $\Phi(d_n) = m!d_{\frac{n}{m!}}$ and $\Phi(I(n)) = m!I(\frac{n}{m!}), \forall n \in \mathbb{Z}$.

Then $V^{(m)}$ can be viewed as an HVir module via Φ and is isomorphic to $V(\alpha', \beta'; F')$ as modules over HVir for suitable $\alpha', \beta', F' \in \mathbb{C}$, by Theorem 2.3. Then there is a basis $\{v'_q \mid q \in \text{supp } V^{(m)}\}$ of $V^{(m)}$ satisfying:

$$(m!d_{\frac{i}{m!}})v'_{\frac{k}{m!}} = (\alpha' + k + i\beta')v'_{\frac{k+i}{m!}}, \quad (m!I(\frac{i}{m!}))v'_{\frac{k}{m!}} = F'v'_{\frac{k+i}{m!}}, \quad \forall \frac{i}{m!}, \frac{k}{m!} \in \text{supp } V^{(m)}.$$

That is:

$$d_{\frac{i}{m!}} v'_{\frac{k}{m!}} = (\frac{\alpha'}{m!} + \frac{k}{m!} + \frac{i}{m!} \beta') v'_{\frac{k+i}{m!}}, \quad I(\frac{i}{m!}) v'_{\frac{k}{m!}} = \frac{F'}{m!} v'_{\frac{k+i}{m!}},$$

or that,

$$d_p v'_q = (\frac{\alpha'}{m!} + q + p\beta') v'_{p+q}, \quad I(p) v'_q = \frac{F'}{m!} v'_{p+q}, \quad \forall p, q \in \text{supp } V^{(m)}.$$

Taking $p = 0$ and $q \in \text{supp } V^{(m-1)} \subset \text{supp } V^{(m)}$, we then have that $d_0 v'_q = (\frac{\alpha'}{m!} + q) v'_q$ and $I(0) v'_q = \frac{F'}{m!} v'_q$, which indicates that $\alpha' = m! \alpha$ and $F' = m! F$.

Assume that $v'_q = c_q v_q$, $\forall q \in \text{supp } V^{(m-1)}$. Compare $I(p) v'_q = F v'_{p+q}$ and $I(p) v_q = F v_{p+q}$. We see that c_p is independent to p . Then we may choose $v'_p = v_p$ for all $p \in \text{supp } V^{(m-1)}$. By the remark in Sect.2 we see that $\beta' = \beta$. Denote $v_q = v'_q$ for all $q \in \text{supp } V^{(m)} \setminus \text{supp } V^{(m-1)}$. Thus the basis $\{v_q | q \in \text{supp } V^{(m)}\}$ is an extension of $\{v_p | p \in \text{supp } V^{(m-1)}\}$ such that $d_p v_q = (\alpha + q + p\beta) v_{p+q}$ and $I(p) v_q = F v_{p+q}$, $\forall p, q \in \text{supp } V^{(m)}$.

By induction, we can have a basis $\{v_q | q \in \text{supp } V\}$ for V such that $d_p v_q = (\alpha + q + p\beta) v_{p+q}$, $I(p) v_q = F v_{p+q}$, $\forall p, q \in \text{supp } V$ and $C_D V = C_I V = C_{DI} V = 0$. That is, $V \cong V'(\alpha, \beta; F)$. This completes the proof. \square

Recall from [K] that the **rank** of an additive subgroup G of \mathbb{C} , denoted by $\text{rank}(G)$, is the maximal number r with $g_1, \dots, g_r \in G \setminus \{0\}$ such that $\mathbb{Z}g_1 + \dots + \mathbb{Z}g_r$ is a direct sum. If such an r does not exist, we define $\text{rank}(G) = \infty$.

Now we can see that the proof of Theorem 3.3 is also valid for G being an infinitely generated additive subgroup of \mathbb{C} of rank 1. Thus, we have the following:

Theorem 3.3. *Let G be an infinitely generated additive subgroup of \mathbb{C} with rank 1. Then any nontrivial irreducible Harish-Chandra module over $\text{HVir}[G]$ must be isomorphic to $V'(\alpha, \beta; F)$ for suitable $\alpha, \beta, F \in \mathbb{C}$. \square*

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